Stats 369. Homework 6.

Exercise 2.4.

(a)

We copy here the the partition function of the SK model

$$Z_n(\beta,\lambda) = \sum_{\sigma\in\{\pm1\}^n} \exp\left[\frac{\beta\lambda}{2n}\langle \boldsymbol{x}_0, \sigma\rangle^2 + \langle W, \sigma\sigma^{\mathsf{T}}\rangle\right]$$

where $W \sim \mathrm{GOE}(n)$.

We have

$$\mathbb{E}[Z_n^r] = \int \exp[nS(Q)DQ] \doteq \exp[n \max_Q S(Q)],$$

where

$$S(Q) = \frac{\beta^2 r}{4} - \frac{\beta^2}{2}\sum_{1\le a<b\le r} Q_{a,b}^2 - \frac{\beta\lambda}{2}\sum_{a=1}^r Q_{0,a}^2 + \log z_r(Q_{1\mathrm{RSB}})$$

and

$$\begin{aligned} z_r(Q_{1\mathrm{RSB}}) =& \frac{1}{2}\sum_{\sigma^0,\dots,\sigma^r\in\{\pm1\}} \exp[\beta\lambda b\sum_{a=1}^r \sigma^0\sigma^a + \beta^2\sum_{1\le a<b\le r} Q_{a,b}\sigma^a\sigma^b] \\ =& \sum_{\sigma^1,\dots,\sigma^r\in\{\pm1\}} \exp\left[\beta\lambda b\sum_{a=1}^r \sigma^a + \frac{\beta^2}{2}\left[q_0\Big(\sum_{a=1}^r\sigma^a\Big)^2 + (q_1-q_0)\sum_{l_1}\Big(\sum_{a\in B_{l_1}}\sigma^a\Big)^2 - q_1 r\right]\right] \\ =& \exp\left(-\frac{\beta^2 q_1 r}{2}\right)T, \end{aligned}$$

where

$$T = \mathbb{E}\left[\sum_{\sigma^1,\dots,\sigma^r\in\{\pm1\}} \exp\left[\beta(\lambda b + \sqrt{q}_0 g_0)\cdot\Big(\sum_{a=1}^r\sigma^a\Big) + \beta\sqrt{q_1-q_0}\sum_{l_1} g_{l_1}\Big(\sum_{a\in B_{l_1}}\sigma^a\Big)\right]\right].$$

Taking parallel calculations with $\lambda = 0$ case, we have

$$\Psi_{1\mathrm{RSB}}(q_0,q_1,m_0,b;\beta,\lambda) = \frac{\beta^2}{4}(1-2q_1+m_0q_0^2+(1-m_0)q_1^2) - \frac{\beta\lambda}{2}b^2 + F(q_0,q_1,m_0,b;\beta,\lambda),$$

where

$$F(q_0,q_1,m_0,b;\beta,\lambda) = \frac{1}{m_0}\mathbb{E}_{g_0}[\log\left(\mathbb{E}_{g_1}\left[\left(2\cosh\beta(\lambda b + \sqrt{q}_0 g_0 + \sqrt{q_1-q_0}g_1)\right)^{m_0}\right]\right)].$$

(b)

First we consider $q_{0*} = 0$. For a given m_0, the stationery point of the above expression gives

$$b = \frac{\mathbb{E}_{g_1}[(\cosh\beta(\lambda b + \sqrt{q}_1 g_1))^{m_0}\tanh\beta(\lambda b + \sqrt{q}_1 g_1))]}{\mathbb{E}_{g_1}[(\cosh\beta(\lambda b + \sqrt{q}_1 g_1))^{m_0}]},$$

$$\beta m_0\sqrt{q_1} = \frac{\mathbb{E}_{g_1}[(\cosh\beta(\lambda b + \sqrt{q}_1 g_1))^{m_0}\tanh\beta(\lambda b + \sqrt{q}_1 g_1))g_1]}{\mathbb{E}_{g_1}[(\cosh\beta(\lambda b + \sqrt{q}_1 g_1))^{m_0}]}.$$

For a given m, we use the fixed point iterations to compute $b(m)$ and $q_1(m)$, and we minimize $\Psi_{1\mathrm{RSB}}(q_0 = 0, q_1(m), m, b(m))$ with respect to m.

The Matlab program is as following.

```matlab
clc;clear;close all;

beta = 1;
lambdaset = 1e-5:0.1:3;
for iterlambda = 1:length(lambdaset)
lambda = lambdaset(iterlambda);
mset = 1e-5:0.01:0.999;
eps = 1e-5;

for iterm = 1:length(mset)
    iterm
    m = mset(iterm);
    b = 0.5;
    q = 0.5;
    bpre = b;
    qpre = q;
    D = 100;
    maxiter = 100000;
    for iter = 1:maxiter
        f = @(x) tanh(beta * (lambda * b + sqrt(q) * x)) .* cosh(beta * (lambda * b + sqrt(q) * x)).^m .* exp(-x.^2/2) /sqrt(2*pi);
        g = @(x) tanh(beta * (lambda * b + sqrt(q) * x)) .* cosh(beta * (lambda * b + sqrt(q) * x)).^m .* exp(-x.^2/2) /sqrt(2*pi) .*x;
        h = @(x) cosh(beta * (lambda * b + sqrt(q) * x)).^m .* exp(-x.^2/2) / sqrt(2*pi);
        b = integral(f, -D, D)/integral(h, -D, D);
        q = min((integral(g, -D, D)/integral(h, -D, D)/beta/m)^2,1);
        if abs(qpre - q) < eps & abs(bpre - b) < eps
            break
        end
        bpre = b;
        qpre = q;
    end
    Phi = beta^2 /4 * (1 - 2*q + (1-m) * q^2) - beta * lambda/2 * b^2 + 1/m * log(2^m * integral(h, -D, D));
    Phires(iterm) = Phi;
    bres(iterm) = b;
    qres(iterm) = q;
end

[~, id] = min(Phires);
b_lambda(iterlambda) = bres(id);
```

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end

save(strcat('beta_', int2str(beta),'.mat'));
```

(c)

We know that $b_*(\beta,\lambda) = M(\beta,\lambda)$. The curves for $b_*(\beta,\lambda)$ as a function of λ with different choice of β is given as following.

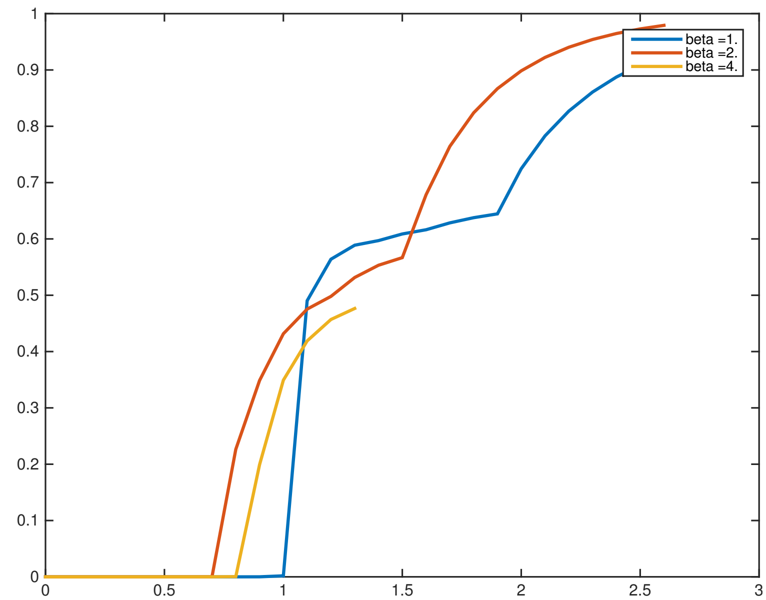

Figure 1: $M(\beta,\lambda)$.